# TOPOLOGY - III, EXERCISE SHEET 11

**Exercise 1.** *Examples of CW complexes.*
Recall the definition of a CW complex and show that the following topological spaces exhibit a CW complex structure:

(1) $S^n$
(2) All graphs $\Gamma$.
(3) $(T^2)^{\#n}$
(4) $(\mathbb{RP}^2)^{\#n}$
(5) $\mathbb{RP}^n$
(6) $\mathbb{CP}^n$.

**Exercise 2.** *Simplicial Homology in $\mathbb{Z}/2\mathbb{Z}$ - coefficients.*
Recall from exercise 9 of sheet 3 that calculating the simplicial homology of surfaces was computationally challenging, before we had tools like the Mayer-Vietoris sequence. However, computing homology in $\mathbb{Z}/2\mathbb{Z}$ coefficients makes computations much simpler in general.

Using the $\Delta-$complex structure from exercise 9 of sheet 3, compute the simplicial homology of $\mathbb{RP}^2$ and $T^2$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

**Exercise 3.** *Universal Coefficient theorem*
Use the universal coefficient theorem in homology to calculate all homology groups of the following spaces with coefficients in the prescribed abelian groups:

(1) $H_*(S^n; G)$ for all abelian groups $G$.
(2) $H_*(T^2; G)$ for $G = \mathbb{Q}, \mathbb{R}, \mathbb{Z}/2\mathbb{Z}$.
(3) $H_*(\mathbb{RP}^2; G)$ for $G = \mathbb{Q}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$.

**Exercise 4.** *Independent nowhere vanishing vector fields on $S^3$ via the quaternions.*
Recall that the real vector space $\mathbb{R}^4$ can be endowed with the structure of an associative algebra called the quaternion algebra $Q$. Here $Q := \{a + bi + cj + dk \mid a, b, c, d \in R\}$ and $1, i, j, k$ follow the relations as in the quaternion group $Q_8$. Show that multiplication by $i, j, k$ can be used to define three nowhere vanishing vector fields on $S^3$, which are linearly independent at every point on $S^3$.